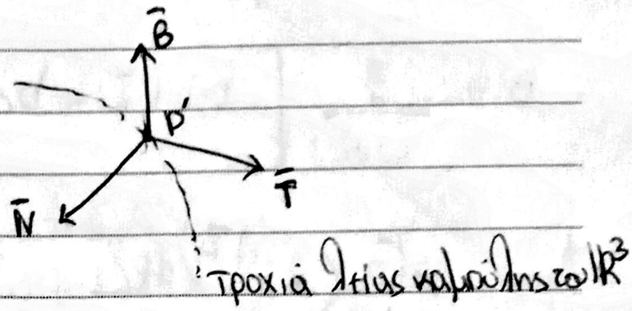
04/03/2018

Μάθημα 4ο

$$\bar{T} = \frac{\bar{v}}{|\bar{v}|}, \quad \bar{N} = \frac{1}{\kappa}\frac{d\bar{T}}{ds}, \quad \kappa = \left|\frac{d\bar{T}}{ds}\right|, \quad \bar{B} = \bar{T} \times \bar{N}$$

$$\tau = -\frac{d\bar{B}}{ds} \cdot \bar{N}$$



$S = S(t) : \bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$

$\bar{v} = \frac{d\bar{r}}{dt}$: ο ρυθμός μεταβολής του διανύσματος θέσης.

$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2 r}{dt^2}$: η μεταβολή της ταχύτητας ως προς τον χρόνο

Στο Νέο Σύστημα Συντεταγμένων (φυσικό: κινείται)

$$T = \frac{\bar{v}}{|\bar{v}|} \Rightarrow \bar{v} = |\bar{v}| \cdot \bar{T} = |\bar{v}|\bar{T} + 0\bar{N} + 0\bar{B}$$

$$\frac{d(f \cdot g)}{dt} = \frac{df}{dt}g + f\frac{dg}{dt}$$

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt}(|\bar{v}| \cdot \bar{T}) = \frac{d}{dt}\left(\frac{ds}{dt} \cdot \bar{T}\right) = \frac{d^2 s}{dt^2}\bar{T} + \frac{ds}{dt}\frac{d\bar{T}}{dt} =$$

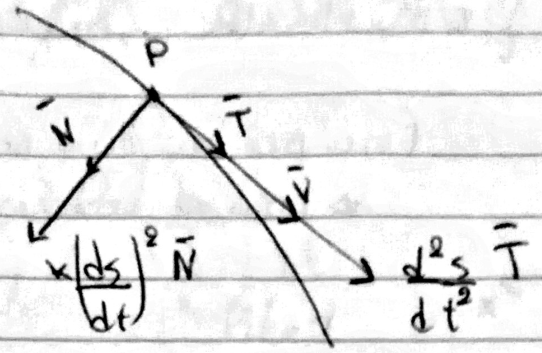
$$= \frac{d^2 s}{dt^2}\bar{T} + \frac{ds}{dt}\left(\frac{d\bar{T}}{ds} \cdot \frac{ds}{dt}\right) =$$

$$= \frac{d^2 s}{dt^2}\bar{T} + \left(\frac{ds}{dt}\right)^2 \kappa\bar{N} = \frac{d^2 s}{dt^2} \cdot \bar{T} + \kappa\left(\frac{ds}{dt}\right)^2 \bar{N}$$

επιτρόχια (a_N) κεντρομόλος (a_T)

2 συνιστώσες

η επιτάχυνση



μεγαλύτερο της μονάδας

Αφού $\bar v = |\bar v|\cdot \bar T$, $\bar a = \frac{d^2s}{dt^2}\bar T + \kappa\left(\frac{ds}{dt}\right)^2 \bar N$, $|\bar v| = \frac{ds}{dt}$

Υπολογίζω το εξωτερικό γινόμενο :

$$\bar v\times\bar a = \left(\frac{ds}{dt}\cdot\frac{d^2s}{dt^2}\right)\underbrace{\bar T\times\bar T}_{=\bar 0} + \kappa\frac{ds}{dt}\left(\frac{ds}{dt}\right)^2\bar T\times\bar N \Rightarrow$$

$$\bar v\times\bar a = \kappa\left(\frac{ds}{dt}\right)^3\underbrace{\bar T\times\bar N}_{=\bar B}$$ κάθετο στο επίπεδο που σχηματίζουν τα $\bar T, \bar N$

$$|\bar v\times a| = \kappa\left(\frac{ds}{dt}\right)^3\cdot|\bar B| \Rightarrow |\bar v\times\bar a| = \kappa\cdot|\bar v|^3\cdot 1 \Rightarrow$$

$$\Rightarrow \kappa = \frac{|\bar v\times\bar a|}{|\bar v|^3}$$ άλλος τρόπος έκφρασης της καμπυλότητας

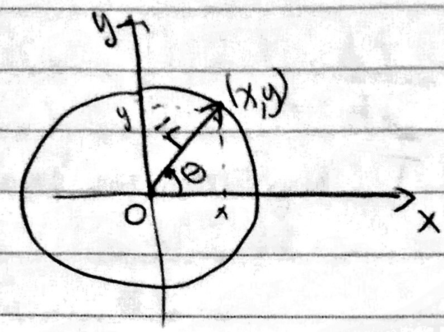
αντίστοιχα για την στρέψη :

$\bar B = \dddot{x}\hat i + \dddot{y}\hat j + \dddot{z}\hat k = \frac{d}{dt}\ddot{x}\hat i + \frac{d}{dt}\ddot{y}\hat j + \frac{d}{dt}\ddot{z}\hat k$

$\bar v\cdot(\bar a\times\bar B)$ =

$$\tau = \frac{\begin{vmatrix}\dot x & \dot y & \dot z\\ \ddot x & \ddot y & \ddot z\\ \dddot x & \dddot y & \dddot z\end{vmatrix}}{|\bar v\times\bar a|^2} \quad , \quad \bar v\times\bar a \neq \bar 0$$

Στο επίπεδο: Έστω $D \subseteq \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, διδιάστατο καρτεσιανό

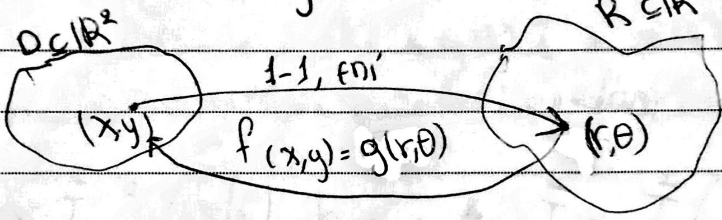


Έστω ότι δουλεύουμε σε κυκλική κίνηση.
→ πάω σε πολικές: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$r = |\vec{r}|$

$f(x) \longmapsto g(y)$

$x \longmapsto y$



$$\frac{\partial x}{\partial r} = \cos\theta \quad , \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta \quad , \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

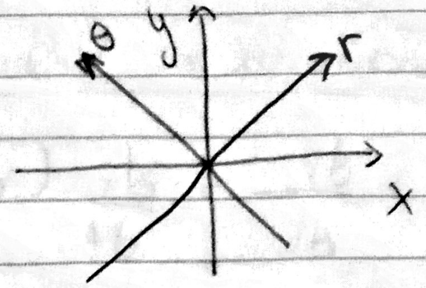
$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$ πίνακας στροφής

μετρικές του μετασχηματισμού

" J Ιακωβιανός → 1ης τάξης πληροφορία

$|J| = r$, αντίστροφη ορίζουσα $-1/r$
ορίζουσα

$f(x,y) \to (\varepsilon, j)$, $x = x(\varepsilon, j)$



$$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial x}{\partial}\frac{\partial}{\partial t}$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial y}{\partial j}\frac{\partial j}{\partial t}$$

Διανυσματικές Μονάδες στις πολικές

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} , \quad \bar{r} = x\bar{i} + y\bar{j}$$

$$\frac{\partial \bar{r}}{\partial r} = \frac{\partial x}{\partial r}\bar{i} + \frac{\partial y}{\partial r}\bar{j} = \cos\theta\,\bar{i} + \sin\theta\,\bar{j}$$

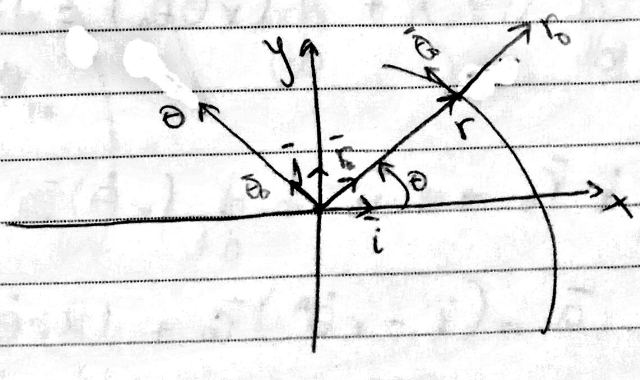
$$\frac{\partial \bar{r}}{\partial \theta} = \frac{\partial x}{\partial \theta}\bar{i} + \frac{\partial y}{\partial \theta}\bar{j} = -r\sin\theta\,\bar{i} + r\cos\theta\,\bar{j}$$

$$\bar{e}_2 = \bar{\theta}_0 = \frac{\frac{\partial \bar{r}}{\partial \theta}}{\left|\frac{\partial \bar{r}}{\partial \theta}\right|} = \frac{-r\sin\theta\,\bar{i} + r\cos\theta\,\bar{j}}{r} = -\sin\theta\,\bar{i} + \cos\theta\,\bar{j}$$

$$\bar{e}_1 = \bar{r}_0 = \frac{\frac{\partial \bar{r}}{\partial r}}{\left|\frac{\partial \bar{r}}{\partial r}\right|} = \frac{\cos\theta\,\bar{i} + \sin\theta\,\bar{j}}{1} = \cos\theta\,\bar{i} + \sin\theta\,\bar{j}$$

$\bar{r}_0 = \cos\theta\,\bar{i} + \sin\theta\,\bar{j}$

$\bar{\theta}_0 = -\sin\theta\,\bar{i} + \cos\theta\,\bar{j}$



Ταχύτητα σε Πολικές (στο επίπεδο)

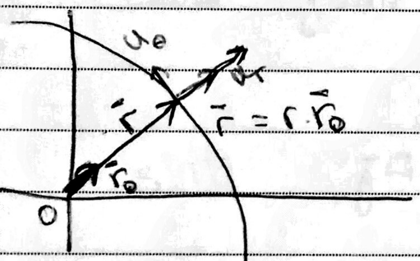
$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{d}{dt}(r\cdot\bar{r}_0) = \frac{dr}{dt}\bar{r}_0 + r\frac{d\bar{r}_0}{dt} = \dot{r}\cdot\bar{r}_0 + r\cdot\dot{\bar{r}}_0$$

- $\bar{r} = |\bar{r}|\cdot\bar{r}_0 = r\cdot\bar{r}_0$

- $\dot{\bar{r}}_0 = \frac{d}{dt}(\cos\theta\,\bar{i} + \sin\theta\,\bar{j}) = -\sin\theta\,\frac{d\theta}{dt}\bar{i} + \cos\theta\,\frac{d\theta}{dt}\bar{j} =$
  $= -\dot{\theta}\sin\theta\,\bar{i} + \dot{\theta}\cos\theta\,\bar{j} = \dot{\theta}\,\bar{\theta}_0$

Άρα: $\bar{v} = \dot{r}\,\bar{r}_0 + r\dot{\theta}\,\bar{\theta}_0 = u_r\bar{r}_0 + u_\theta\cdot\bar{\theta}_0$

όπου $\begin{cases} u_r = \dot{r} & \text{, ακτινική συνιστώσα} \\ u_\theta = r\cdot\dot{\theta} & \text{, εγκάρσια συνιστώσα} \end{cases}$



Επιτάχυνση σε πολικές

$$\bar{a} = \frac{d^2\bar{r}}{dt^2} = \frac{d\bar{v}}{dt} = \frac{d}{dt}(\dot{r}\bar{r}_0 + r\dot{\theta}\bar{\theta}_0) =$$

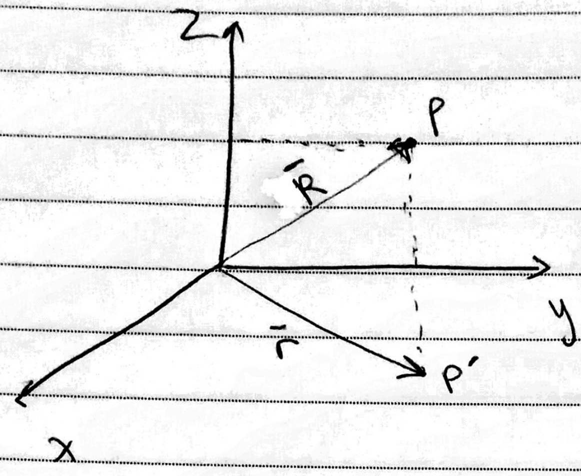
$$= \frac{d}{dt}(\dot{r}\cdot\bar{r}_0) + \frac{d}{dt}(r\dot{\theta}\bar{\theta}_0) = \frac{dr}{dt}\bar{r}_0 + r\frac{d\bar{r}_0}{dt} + \frac{d}{dt}(r\dot{\theta})\bar{\theta}_0 + r\dot{\theta}\frac{d\bar{\theta}_0}{dt} =$$

$$= \ddot{r}\,\bar{r}_0 + r\cdot\dot{\bar{r}}_0 + \frac{d}{dt}(r\dot{\theta})\bar{\theta}_0 + r\dot{\theta}\dot{\bar{\theta}}_0 \Rightarrow$$

$$\Rightarrow \bar{a} = \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{\text{ακτινική}}\bar{r}_0 + \underbrace{(2\dot{r}\dot{\theta} + r\ddot{\theta})}_{\text{εγκάρσια}}\bar{\theta}_0$$

έχει λάθος κάπου

Κυλινδρικές Συντεταγμένες



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\bar{R} = x\bar{i} + y\bar{j} + z\bar{k} = (r\cos\theta)\bar{i} + (r\sin\theta)\bar{j} + z\bar{k}$$

Ορίζω τα μοναδιαία διανύσματα :

$$\bar{e}_1 = \bar{r}_0 = \frac{\frac{\partial \bar{r}}{\partial r}}{\left|\frac{\partial \bar{r}}{\partial r}\right|} = \cos\theta\bar{i} + \sin\theta\bar{j}$$

$$\bar{e}_2 = \bar{\theta}_0 = \frac{\frac{\partial \bar{r}}{\partial \theta}}{\left|\frac{\partial \bar{r}}{\partial \theta}\right|} = -\sin\theta\bar{i} + \cos\theta\bar{j}$$

$$\bar{e}_3 = \bar{z}_0 = \frac{\frac{\partial \bar{r}}{\partial z}}{\left|\frac{\partial \bar{r}}{\partial z}\right|} = 0\bar{i} + 0\bar{j} + \underbrace{1\cdot\bar{k}}_{\bar{z}_0}$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|J| = r$$